

# Linear spectral transformations of Carathéodory functions

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## Abstract

In this paper we present some recent results concerning linear spectral transformations of Carathéodory functions. More precisely, given two Carathéodory functions related by a linear spectral transformation, we study the relation between the corresponding moment functionals and, in the positive definite case, the relation between the measures.

We will see that rational modifications of functionals are included in the linear spectral transformations. However, we will show that there exist a huge class of linear spectral transformations which are not given by rational modifications of functionals. Indeed, we will characterize those linear spectral transformation which come from a rational modification.

In the general case we will discuss the relation between the functionals involved in a linear spectral transformation, which allows us to identify the difficulties to connect the related functionals.

Actually, several examples will show how amazing can be the relationships between the moment functionals associated with a linear spectral transformation.

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## 1 Introduction

The study of modifications of hermitian linear functionals has been considered by several authors. Among others, the contributions [1, 2, 3, 4] are remarkable. In some cases the relation between the Carathéodory functions is known. One of the cases that has been studied more profusely is the situation in which the Carathéodory functions are related by a rational transformation, (see [5]). We will study a special class of these transformation, the so-called linear spectral transformations. We will perform the analysis in the general case of (non necessarily positive-definite, neither quasi-definite) hermitian functionals.

Before discussing the results we will introduce some conventions. We will use the notation  $\mathbb{P} = \mathbb{C}[z]$ ,  $\mathbb{P}_* = \mathbb{C}[z^{-1}]$ ,  $\Lambda = \mathbb{P} \cup \mathbb{P}_*$ ,  $\mathbb{P}_n = \text{span}\{1, z, \dots, z^n\}$  and  $\Lambda_{p,q} = \text{span}\{z^p, \dots, z^q\}$ ,  $p \leq q$ ,  $p, q \in \mathbb{Z}$ . If  $p = -\infty$  or  $q = \infty$  we will not understand  $\Lambda_{p,q}$  as a set of finite linear combinations of powers of  $z$ , but as a complex linear space of formal series. We will work with functionals  $u \in \Lambda' = \text{Hom}(\Lambda, \mathbb{C})$ . For any  $u \in \Lambda'$  and  $L \in \Lambda$  we define  $uL \in \Lambda'$  by  $uL[f] = u[Lf]$ . If  $\mu_n = u[z^n]$ ,  $n \in \mathbb{Z}$ , are the moments of  $u$ , we will say that the functional  $u$  is hermitian if  $\mu_{-n} = \overline{\mu_n}$ . In this case  $\mu_0 \in \mathbb{R}$ . We use the notation  $\mathcal{H} = \{u \in \Lambda' \mid u \text{ is Hermitian}\}$ . A particular case of hermitian functionals are those defined by a measure  $\mu$  on the unit circle  $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$  via  $u[f] = \int_{\mathbb{T}} f(z) d\mu(z)$ . The

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functional given by a Dirac delta at a point  $\alpha$  will be denoted by  $\delta_\alpha$ , while the functional defined by the Lebesgue measure on the unit circle will be denoted by  $\text{leb}$ .

For  $u \in \mathcal{H}$  we define the Carathéodory formal series (CS)

$$F(z) = \mu_0 + 2 \sum_{n \geq 1} \mu_{-n} z^n \in \Lambda_{0,\infty}.$$

If  $F$  is summable in a certain domain  $\Omega \subset \mathbb{C}$ , we will refer to  $F$  as a Carathéodory function in  $\Omega$ . The operators  $*$  and  $^{*p}$ ,  $p \in \mathbb{Z}$ , act on a formal series  $H$  –and, in particular, on any Laurent polynomial– as  $H_*(z) = \overline{H(1/\bar{z})}$  and  $H^{*p}(z) = z^p H_*(z)$ . If  $P$  is a polynomial of degree  $p$  we write  $P^{*p} = P^*$ . Given  $u \in \Lambda'$ , we define the Laurent formal series (LS)

$$\mathcal{L}[u](z) = \sum_{-\infty}^{\infty} \mu_{-n} z^n \in \Lambda_{-\infty,\infty},$$

which characterizes completely the functional  $u$ , i.e., there is a one-to-one correspondence

$$\begin{aligned} \mathcal{L} : \Lambda' &\longrightarrow \{LS\}. \\ u &\longmapsto \mathcal{L}[u] \end{aligned}$$

If the functional  $u$  is hermitian, its LS and CS are related by

$$\mathcal{L}[u](z) = \frac{F(z) + F_*(z)}{2}.$$

It is immediate to check that

$$\mathcal{L}[u + v] = \mathcal{L}[u] + \mathcal{L}[v], \quad \mathcal{L}[uL] = \mathcal{L}[u]L_*, \quad \forall u \in \Lambda', L \in \Lambda.$$

## 2 Linear spectral transformations and rational modifications

In what follows,  $u$  and  $v$  will denote two hermitian linear functionals with Carathéodory series  $F$  and  $G$  respectively.

**Definition 2.1.** We will say that  $u$  and  $v$  are related by a linear spectral transformation (LST) if there exist  $L, M \in \Lambda_0 := \Lambda \setminus \{0\}$  and  $C \in \Lambda$  such that  $FL = GM + C$ .

**Definition 2.2.** We will say that  $u$  and  $v$  are related by a rational modification (RM) if there exist  $L, M \in \Lambda_0$  such that  $uL = vM$ .

We are interested in studying the possible connections between two hermitian linear functionals  $u$  and  $v$  whose corresponding Carathéodory functions are related by a LST. In particular we will clarify the relation between RM and LST. Our first result is the following one.

**Proposition 2.3.** *If the linear functionals  $u, v$  are related by a RM, then the corresponding CS are linked by a LST, i.e.,  $\text{RM} \Rightarrow \text{LST}$ .*

*Proof.* Let  $u, v \in \Lambda'$  such that  $uL_* = vM_*$ , with  $L, M \in \Lambda_{p,q}$  ( $p \leq q$ , finite). Equivalently,

$$(F + F_*)L = (G + G_*)M.$$

Taking  $C = FL - GM = G_*M - F_*L$ , then  $FL - GM \in \Lambda_{p,\infty}$ ,  $G_*M - F_*L \in \Lambda_{-\infty,q}$ . In other words,  $C \in \Lambda_{p,q}$  satisfies  $FL = GM + C$ .  $\square$

*Remark 2.4.* The converse of Proposition 2.3 is not true in general. To see this, it is enough to consider  $v = \delta_1$ , whose Carathéodory function is  $G(z) = (1+z)/(1-z)$ ,  $z \in \mathbb{C} \setminus \{1\}$ . The LST given by  $F(z) = G(z)(1-z^2)$  provides a Carathéodory function  $F$  without poles and, consequently, the corresponding orthogonality measure  $\mu$  is absolutely continuous. Besides,  $\mu$  is supported on the whole unit circle. Hence, there is no RM between  $u$  and  $v$  because, if two functionals defined by measures are related by a RM, the limit points of the support of both measures must coincide.

Although not every LST comes from a RM, Proposition 2.3 states that RM constitute an important subclass of LST. Our first aim is to perform a detailed analysis of the RM subclass. The study of the transformations in  $\{\text{LST}\} \setminus \{\text{RM}\}$  will be the objective of Section 4.

First of all we should note that RM and LST are both equivalence relations. Proposition 2.3 states that RM is a finer equivalence relation than LST. To study properly the RM relation it is convenient to separate an equivalence class to avoid certain ambiguities.

**Definition 2.5.** We define the class  $\Delta$  of hermitian linear functionals by

$$\Delta = \{u \in \mathcal{H} \mid \exists L \in \Lambda_0 \text{ s.t. } uL = 0\}.$$

Definition 2.5 means that  $\Delta$  is the equivalence class of the null functional with respect to the equivalence relation RM. The referred ambiguity comes from the fact that every pair  $u, v \in \Delta$  satisfies  $uL = vM = 0$  for some  $L, M \in \Lambda$ . Therefore, the equality  $uL = vMN$  is also true for all  $N \in \Lambda$ . Thus, the correspondence  $L \leftrightarrow M$  is not biunivocal in  $\Delta$ . To avoid these ambiguities we will work with RM in  $\mathcal{H} \setminus \Delta$ .

The condition  $u \in \Delta$  is equivalent to  $uQ = 0$  for some  $Q \in \mathbb{P}$ , which means that the functional  $u$  is a linear combination of  $\delta'$ 's and their derivatives supported on the non-zero roots of  $Q$ . Since the functional  $u$  is hermitian, these  $\delta'$ 's are supported on symmetric points  $\alpha, 1/\bar{\alpha}$  with conjugated masses. Thus, we can choose the minimal polynomial  $Q$  such that  $uQ = 0$  satisfying  $Q = Q^*$ . The minimality of  $Q$  means that  $uQ_1 = 0$  implies that  $Q$  divides  $Q_1$ . In such a case the CS  $F$  of  $u$  is summable as a meromorphic function in  $\mathbb{C}$ ,  $F = P/Q$ ,  $P$  being a polynomial with  $\deg P \leq \deg Q = q$ .

In terms of LST,

$$(F + F_*)Q_* = 0.$$

Thus,

$$z^{-q}FQ + F_*Q_* = 0 \quad \text{and} \quad z^{-q}P + P_* = 0,$$

i.e.,

$$P = -z^q P_* = -P^{*q}.$$

Summarizing, for any  $u \in \Delta$ ,  $F$  is summable as a rational function

$$F = \frac{P}{Q}, \quad Q = Q^*, \quad P^{*\deg Q} = -P.$$

Actually, the above property characterizes the functionals belonging to  $\Delta$ . In fact, from

$$FQ = P$$

and

$$F_*Q_* = P_* = z^{-q}P^{*q} = -z^{-q}P$$

we obtain

$$(F + F_*)Q = 0,$$

i.e.,

$$uQ = 0.$$

Note that  $P(0)/Q(0) \in \mathbb{R}$  due to the hermitian character of  $u$ .

Besides, if  $Q$  is minimal,  $\gcd(P, Q) = 1$ . Indeed, if  $S$  is a common divisor of  $P$  and  $Q$ ,  $S^*$  is also common divisor of  $P$  and  $Q$ . Then,  $P$  and  $Q$  are divisible by some  $T \in \mathbb{P}$  with  $T = T^*$ . The polynomials  $P_1 = P/T$  and  $Q_1 = Q/T$  satisfy  $Q_1^* = Q_1$ ,  $P_1^{*\deg Q_1} = -P_1$  and  $F = P_1/Q_1$ . Thus,  $uQ_1 = 0$ , which contradicts the minimality of  $Q$ .

In what follows we will work with RM in  $\mathcal{H} \setminus \Delta$  to avoid the ambiguities in the relation  $L \leftrightarrow M$  that appear in the class  $\Delta$ .

**Proposition 2.6.** *Let  $u, v \in \mathcal{H}$ . Then, if  $uL = vM$ , the correspondence  $L \leftrightarrow M$  is biunivocal iff  $u, v \notin \Delta$ .*

*Proof.* We have shown that the correspondence  $L \leftrightarrow M$  is not biunivocal if  $u$  and  $v$  lie in the equivalence class  $\Delta$ . On the contrary, if  $u, v \in \mathcal{H} \setminus \Delta$ , then  $vM_1 = uL = vM$  implies  $v(M - M_1) = 0$ , thus  $M = M_1$ .  $\square$

### 3 Minimality of RM and LST

For RM in  $\mathcal{H} \setminus \Delta$  the correspondence  $L \leftrightarrow M$  is biunivocal, but the pair  $(L, M)$  is not unique. Indeed,  $uL = vM$  implies  $uLN = vMN$ ,  $\forall N \in \Lambda$ . Our aim is to choose the simplest pair  $(L, M)$  satisfying  $uL = vM$  for each  $u, v$  in the same RM equivalence class of  $\mathcal{H} \setminus \Delta$ . This means to cancel the common factors of  $L$  and  $M$  when it is possible. However, this cancelation is not always viable as we show in the following example.

**Example 3.1.** Let  $u = \text{leb}$  and  $v = \text{leb} + \delta_1$ . Obviously, the relations

$$u(z^2 - 1) = v(z^2 - 1) \quad \text{and} \quad u(z - 1) = v(z - 1)$$

are satisfied. From the first relation to the second one, a common factor is simplified. However, the second relation is not reducible due to the presence of  $\delta_1$  in  $v$ .

We are interested in minimal expressions which are not reducible by simplifying common factors of  $L$  and  $M$ . When  $u, v \in \Delta$ , the minimal expressions are not unique:  $\delta_1(z - 1) = \delta_{-1}(z + 1)$  and  $\delta_1(z^2 - 1) = \delta_{-1}(z + 1)$  are both not reducible. In contrast, we will see that there is an essentially unique minimal expression of a RM if  $u, v \notin \Delta$ .

For this purpose it is convenient to consider the set of all pairs  $(L, M)$  behind a given RM.

**Definition 3.2.** Given  $u, v \in \mathcal{H}$  we define

$$\mathbb{I} = \mathbb{I}(u, v) = \{(L, M) \in \Lambda_0 \times \Lambda_0 \mid uL = vM\}.$$

Note that  $U = \{\alpha z^k \mid \alpha \in \mathbb{C}^*, k \in \mathbb{Z}\}$  is the group of units of the ring  $\Lambda$ . Thus, we can define a partial order relation in  $\mathbb{I}$  by

$$(L_0, M_0) \leq (L_1, M_1) \iff N(L_0, M_0) = (L_1, M_1), \quad N \in \Lambda,$$

and

$$(L_0, M_0) \equiv (L_1, M_1) \iff N(L_0, M_0) = (L_1, M_1), \quad N \in U.$$

With respect to this order relation,  $\mathbb{I}$  is an inductive set. Since  $\mathbb{I}$  is bounded from below by  $(0, 0)$ ,  $\mathbb{I}$  has at least a minimal element. In what follows we will suppose that  $\mathbb{I} \neq \emptyset$ , i.e.,  $u, v$  are related by a RM.

Our first result about minimality uses the notion of gcd in  $\Lambda$ , which is unique up to factors in  $U$ . Indeed the division algorithm translates from  $\mathbb{P}$  to  $\Lambda = U\mathbb{P}$ , and the gcd becomes unique in  $\Lambda/U \cong \mathbb{P}/\mathbb{C}^*$ .

**Theorem 3.3.** *Let  $u, v \in \mathcal{H} \setminus \Delta$  be related by a RM. Then,  $\mathbb{I} = \mathbb{I}(u, v)$  has a minimal element  $(L, M)$  which is unique up to units of  $\Lambda$ . Besides,  $\mathbb{I} = \Lambda_0(L, M) := (\Lambda_0 L, \Lambda_0 M)$ .*

*Proof.* Let  $(L_0, M_0) = (L, M)$  be a minimal element of  $\mathbb{I}$  and  $(L_1, M_1) \in \mathbb{I}$ . Applying the Euclidean algorithm in  $\Lambda$ ,

$$L_k = L_{k+1}Q_{k+1} + L_{k+2}, \quad k = 0, 1, \dots, m-2,$$

with  $Q_{k+1} \in \Lambda$  and  $L_m \in \gcd(L_0, L_1)$ . Therefore,

$$uL_k = uL_{k+1}Q_{k+1} + uL_{k+2} = vM_k = vM_{k+1}Q_{k+1} + uL_{k+2}.$$

Hence

$$uL_{k+2} = v(M_k - M_{k+1}Q_{k+1}) = vM_{k+2}.$$

After  $m-2$  steps, we arrive at  $uL_m = vM_m$ . However, for  $i = 0, 1$ ,  $L_i = P_i L_m$  with  $P_i \in \Lambda$ , thus  $uL_i = uP_i L_m = vP_i M_m$  and  $(L_i, M_i) = P_i(L_m, M_m)$ .

Since  $(L_0, M_0)$  is minimal, then  $P_0 \in U$ . Hence  $(L_1, M_1) = P_1 P_0^{-1}(L_0, M_0)$  and  $\mathbb{I} = \Lambda(L, M)$ . Besides,  $(L_1, M_1)$  minimal leads to  $P_1 \in U$ , so  $(L_1, M_1) \equiv (L_0, M_0)$ .  $\square$

The minimal representation of RM has an additional advantage.

**Corollary 3.4.** *If  $u, v \in \mathcal{H} \setminus \Delta$  and  $uL = vM$  is minimal, then  $(L_*, M_*) \equiv (L, M)$ , i.e., there exist  $\alpha \in \mathbb{C}^*$  and  $p \in \mathbb{Z}$  such that  $(L^{*p}, M^{*p}) = \alpha(L, M)$ .*

*Proof.* Since  $uL = vM$  and  $u, v \in \mathcal{H}$  we have that  $uL_* = vM_*$ . The minimality ensures the existence of  $N \in \Lambda$  such that  $(L_*, M_*) = N(L, M)$ , therefore  $(L, M) = N_*(L_*, M_*)$  and the minimality of  $(L, M)$  implies  $N \in U$ .  $\square$

Multiplying by a suitable factor in  $\mathbb{C}^*$ , the minimal relation  $uL = vM$ , we can find a minimal pair such that  $(L, M) = (L^{*p}, M^{*p})$  for some  $p \in \mathbb{Z}$ . This kind of relation also holds for any other (non-minimal) pair  $(L_1, M_1) = N(L, M)$  as far as  $N_* = z^k N$  for some  $k \in \mathbb{Z}$ . A pair satisfying this condition will be called a symmetric pair. Due to Corollary 3.4, the existence of minimal representations ensures the existence of symmetric pairs.

The use of symmetric pairs allows us to characterize easily those LST which represent RM.

**Corollary 3.5.** *Let  $u, v \in \mathcal{H} \setminus \Delta$ , with  $F, G$  the corresponding CS, and  $L, M \in \Lambda$  such that  $(L^{*p}, M^{*p}) = (L, M)$ . Then,  $uL = vM$  iff  $FL = GM + C$ , where  $C^{*p} = -C$ .*

*Proof.* From Proposition 2.3,  $C = FL - GM = G_*M - F_*L$ , hence

$$C^{*p} = z^p C_* = z^p (F_* L_* - G_* M) = F_* L - G_* M = -C.$$

Conversely, suppose  $FL = GM + C$ , with  $(L^{*p}, M^{*p}) = (L, M)$  and  $C^{*p} = -C$ . Then,

$$FL = GM + C, \quad F_* L^{*p} = G M^{*p} + C^{*p},$$

and,

$$(F + F_*)L = (G + G_*)M.$$

The identification of the LS  $(F + F_*)$  and  $(G + G_*)$  with  $u$  and  $v$ , respectively, gives  $uL_* = vM_*$  or, equivalently,  $uL = vM$ .  $\square$

Example 3.2 shows that it is not always possible to simplify the common factors in a RM. The corresponding LST for this example,  $F(z^2 - 1) = G(z^2 - 1) + (z + 1)^2$ , suggests that the common factors of  $L, M$  that we can eliminate in a relation  $uL = vM$ , are only those which are also common to  $C$  in the corresponding LST  $FL = GM + C$ . This conjecture will be proved in the following theorem which characterizes the minimal representation of a RM in terms of the corresponding LST.

**Theorem 3.6.** *Let  $u, v \in \mathcal{H} \setminus \Delta$  and  $L, M \in \Lambda$  such that  $(L^{*p}, M^{*p}) = (L, M)$ . Then,  $uL = vM$  is minimal iff the corresponding LST  $FL = GM + C$  is such that  $1 \in \gcd(L, M, C)$ .*

*Proof.* Corollary 3.5 ensures the existence of  $C \in \Lambda$  such that  $FL = GM + C$ , and  $C^{*p} = -C$ . This fact together with the symmetry of the pair  $(L, M)$  ensures the existence of a representative  $P \in \gcd(L, M, C)$  such that  $P \in \mathbb{P}$  and  $P^* = P$ . Denoting  $(L_1, M_1, C_1) = (L, M, C)/P$ , we have  $(L_1^{*q}, M_1^{*q}, C_1^{*q}) = (L_1, M_1, -C_1)$  for some  $q \in \mathbb{Z}$ . Corollary 3.5 implies  $uL_1 = vM_1$ . Therefore, the minimality of  $uL = vM$  requires  $\deg P = 0$ .

Conversely, if  $L, M, C$  are coprime and  $uL = vM$  is not minimal, there exists  $P \in \mathbb{P}$  dividing  $L, M$  with  $P^* = P$ . Let  $(L_1, M_1) = (L, M)/P$ . Then,  $(L_1^{*q}, M_1^{*q}) = (L_1, M_1)$  for some  $q \in \mathbb{Z}$ , and  $uL_1 = vM_1$ . Thus, Corollary 3.5 ensures the existence of  $C_1 \in \Lambda$  such that  $FL_1 = GM_1 + C_1$  and, consequently,  $C_1 = C/P$ . Since  $L, M, C$  are coprime,  $\deg P = 0$ .  $\square$

Theorem 3.6 shows how the LST help us to know when a RM is expressed in a minimal way. In what follows we study the minimality of the LST, i.e.,

$$FL = GM + C, \quad 1 \in \gcd(L, M, C).$$

As in the case of the equivalence relation RM, there is an equivalence class for LST where certain ambiguities take place: the LST equivalence class of the null functional.

**Definition 3.7.** We define the class of rational functionals as the set

$$\text{Rat} := \{u \in \mathcal{H} \mid \exists L \in \Lambda_0, N \in \Lambda \text{ s.t. } FL = N\}.$$

Obviously, Rat are the set of functionals whose CS are summable as rational functions. Hence  $\Delta \subset \text{Rat}$ . In the following proposition we describe the functionals belonging to the Rat class.

**Proposition 3.8.** *Let  $u \in \mathcal{H}$  and  $F$  its Carathéodory function. Then,  $u \in \text{Rat}$  iff  $\exists (L, M) \in \Lambda_0 \times \Lambda$  such that  $uL = \text{leb } M$ .*

*Proof.* The functionals  $u \in \Delta$  correspond to  $M = 0$ . If  $u \in \text{Rat} \setminus \Delta$ , there exist  $P, Q \in \mathbb{P}$  such that  $FQ = P$ . Since  $u$  is hermitian, the poles of its Carathéodory function must be symmetric with respect to  $\mathbb{T}$ . Thus, we can choose  $Q = Q^*$ ,  $\deg Q = q$ . Consequently, from

$$FQ = \frac{P + P^{*q}}{2} + \frac{P - P^{*q}}{2} = M + C,$$

where  $M = \frac{P + P^{*q}}{2} = M^{*q}$  and  $C = \frac{P - P^{*q}}{2} = -C^{*q}$ , and Corollary 3.5, we find that  $uQ = \text{leb } M$ .

Conversely, if  $uL = \text{leb } M$ , the hermiticity of  $u$  allows us to assume without loss of generality that  $L = L^{*p}$ ,  $M = M^{*p}$ , for some  $p \in \mathbb{Z}$ . Corollary 3.5 ensures the existence of  $C \in \Lambda$  such that  $C = -C^{*p}$  and  $FL = M + C = N$ .  $\square$

Given the LST  $FL = GM + C$ , the equality  $FLN = GMN + CN$  is also a LST for all  $N \in \Lambda$ . In contrast to RM, the LST are reducible to expressions where  $L, M, C$  are coprime, which will be the minimal LST. As a last question for this section, we ask ourselves when a minimal LST is unique (up to units of  $\Lambda$ ) for a pair of hermitian functionals  $u, v$ . The answer is given by the next proposition.

**Proposition 3.9.** *Let  $u, v \in \mathcal{H}$  and  $F, G$  their respective CS. Then,  $u, v \notin \text{Rat}$  iff the minimal LST,  $FL = GM + C$ , is unique up to units of  $\Lambda$ .*

*Proof.* Let  $FL = GM + C$  and  $FL_1 = GM_1 + C_1$  be minimal, such that  $(L_1, M_1, C_1) \notin U(L, M, C)$ . Then,

$$FLL_1 = GML_1 + CL_1 = GM_1L + C_1L$$

implies

$$G(ML_1 - LM_1) = C_1L - CL_1.$$

If  $v \notin \text{Rat}$ , both sides of the above equation will be zero, which implies the proportionality between  $(L, M, C)$  and  $(L_1, M_1, C_1)$ .

Conversely, let  $u, v \in \text{Rat}$ ,  $FQ = P$ ,  $GS = R$ , with  $P, Q, R, S \in \mathbb{P}$ . Obviously,  $FQ = GS + C$ , with  $C = R - P$ , is a LST for  $u, v$ . Let  $K \in \Lambda_0$ ,  $M = S(K + 1)$  and  $C_1 = C - RK$ . Then,

$$FQ - GM = G(S - M) + C = -GSK + C_1 + RK = C_1,$$

and thus we obtain a new LST for every  $K$ . Choosing  $K = 1$ , we find that  $FQ = 2GS + C - R$ . In this LST,  $(Q, 2S, C - R)$  is not proportional to  $(Q, S, C)$ .  $\square$

As a direct consequence of the previous results we find that the LST relating two functionals  $u, v \in \mathcal{H} \setminus \text{Rat}$  are generated by the unique minimal one  $FL_0 = GM_0 + C_0$ , i.e.,  $\{(L, M, C) \in \Lambda_0 \times \Lambda_0 \times \Lambda \mid FL = GM + C\} = \Lambda(L_0, M_0, C_0)$ .

Besides, according to Corollary 3.4, Corollary 3.5 and Theorem 3.6, to verify that there is a RM behind a LST, we only need to check that the minimal representation  $FL = GM + C$  of a LST satisfies the symmetry conditions  $(L^{*p}, M^{*p}, C^{*p}) = \alpha(L, M, -C)$  for some  $\alpha \in \mathbb{C}^*$ ,  $p \in \mathbb{Z}$ , i.e.,  $(L_*, M_*, C_*) \in U(L, M, -C)$ .

## 4 General LST

Theorem 3.6 provides the conditions which characterize that a LST,  $FL = GM + C$ , in its minimal form, comes from a RM. In this section we will analyze what happens if we remove these symmetry conditions  $(L_*, M_*, C_*) = U(L, M, -C)$ .

Returning to Example 3.1, we can easily check that the Laurent polynomials  $L = 1$ ,  $M = (1 - z^2)$ ,  $C = 0$ , define the minimal form of the corresponding LST, but do not satisfy these symmetry conditions. This means that Example 3.1 is a case of LST which does not come from a RM.

Although the functionals involved in Example 3.1 are both given by positive measures, we should remark that one of these measures is supported on a single point. The following example shows that there exist positive measures supported on infinitely many points which are related by a LST but not by a RM.

**Example 4.1.** We consider the functional  $u$  associated with the Lebesgue measure  $\mu$  supported on the arc  $\Gamma = \{e^{i\theta} \mid \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$ . Is easy to check that the corresponding moments are

$$\mu_0 = 1; \quad \mu_{2n} = 0, \quad \mu_{2n-1} = \frac{2}{\pi} \frac{(-1)^n}{2n-1}, \quad n \geq 1.$$

Therefore, the CS associated with  $u$  is

$$F(z) = 1 + \frac{4}{\pi} \sum_{n \geq 1} (-1)^n \frac{z^{2n-1}}{2n-1},$$

which is summable for  $|z| < 1$

$$F(z) = 1 + \frac{2i}{\pi} \log \left[ \frac{1+iz}{1-iz} \right],$$

and has real part

$$\text{Re } F(re^{i\theta}) = 1 - \frac{2}{\pi} \pi \arg [(1 + ire^{i\theta})(1 + ire^{-i\theta})].$$

Now, we perform the LST

$$G(z) = zF(z) + \alpha,$$

where we will choose  $\alpha$  so that  $\operatorname{Re}(G(z)) > 0$  for  $|z| < 1$ . This condition ensures that  $G$  is the Carathéodory function of a functional  $v$  given by a positive measure  $\nu$  supported on the unit circle. A straightforward computation yields

$$\begin{aligned} \operatorname{Re} G(re^{i\theta}) &\geq \alpha + r \cos \theta \operatorname{Re} F(re^{i\theta}) \\ &= \alpha + r \cos \theta \left( 1 - \frac{2}{\pi} \arg[(1 + re^{i\theta})(1 + ie^{-i\theta})] \right) \geq \alpha - 2r, \end{aligned}$$

which means that  $G$  is a Carathéodory function for  $\alpha \geq 2$ . The radial limits of  $G$  provide the weight of the corresponding measure  $\nu$ , which is absolutely continuous because  $G$  has the same analyticity behavior as  $F$ . We obtain

$$\nu'(\theta) = \alpha + 2 \cos \theta \chi_\Gamma(\theta) + \frac{1}{\pi} \sin \theta \log \left[ \frac{1 + \sin \theta}{1 - \sin \theta} \right].$$

where  $\chi_\Gamma$  is the characteristic function of the arc  $\Gamma$ .

While  $\mu$  is supported only on the arc  $\Gamma$ , the measure  $\nu$  is supported on the whole unit circle  $\mathbb{T}$ . Consequently, although  $u, v$  are related by a LST, they can not be related by a RM.

Note that the Laurent polynomials in the minimal LST,  $G(z) = zF(z) + 2$ , do not satisfy the symmetry conditions  $(L_*, M_*, C_*) = \mathbf{U}(L, M, -C)$ .

In the previous examples we have shown that the transformations in  $\{\text{LST}\} \setminus \{\text{RM}\}$ , in contrast to RM, do not preserve the support of the absolutely continuous part of the measure. We will refer to this as a *wild behaviour*. In what follows we analyze the origin of this behaviour in the connection  $u \leftrightarrow v$ .

We consider a general LST, i.e., without any conditions over the Laurent polynomials  $L, M, C$ , to find out the roots of the wild behavior. Thus, we consider the LST

$$FL = GM + C, \quad L, M \in \Lambda_0, \quad C \in \Lambda. \quad (1)$$

Without loss of generality, we can choose  $L = L^{*p}$  for some  $p \in \mathbb{Z}$ , multiplying the LST by a suitable factor in  $\Lambda_0$ . From (1),  $F_*L_* = G_*M_* + C_*$ . Equivalently,

$$F_*L = F_*L^{*p} = G_*M^{*p} + C^{*p}. \quad (2)$$

From (1) and (2),

$$\frac{F + F_*}{2} L = \frac{G + G_*}{2} \frac{M + M^{*p}}{2} - \frac{G - G_*}{2i} \frac{M - M^{*p}}{2i} + \frac{C + C^{*p}}{2}.$$

Denoting

$$M^+ = \frac{M + M^{*p}}{2}, \quad M^- = \frac{M - M^{*p}}{2i}, \quad C^+ = \frac{C + C^{*p}}{2},$$

we have

$$\frac{F + F_*}{2} L = \frac{G + G_*}{2} M^+ - \frac{G - G_*}{2i} M^- + C^+. \quad (3)$$

The LS  $\frac{G - G_*}{2i}$  has an associated hermitian functional that we will denote by  $\hat{v}$ . Then, (3) can be written in terms of the functionals  $u, v$  and  $\hat{v}$  as

$$uL_* = v(M^+)_* + \hat{v}(M^-)_* + \operatorname{leb}(C^+)_*. \quad (4)$$

Nevertheless,

$$(M^+)_* = \frac{M_* + z^{-p}M}{2} = z^{-p} \frac{M + M^{*p}}{2} = z^{-p} M^+,$$



and, analogously,  $(M^-)_* = z^{-p}M^-$  and  $(C^+)_* = z^{-p}C^+$ . Multiplying Equation (4) by  $z^p$  gives

$$uL = vM^+ - \hat{v}M^- + \text{leb } C^+, \quad (5)$$

where  $(L, M^+, M^-, C^+) = (L, M^+, M^-, C^+)^{*p}$ . The functional  $\hat{v}$  has the associated LS

$$\frac{G - G_*}{2i} = \frac{1}{i} \left( \sum_{n \geq 1} \nu_{-n} z^n - \sum_{n \geq 1} \nu_n z^{-n} \right) = \sum_{n \neq 0} \hat{\nu}_{-n} z^n,$$

with  $\nu_n = v[z^n]$  and  $\hat{\nu}_n = -i\nu_n = \hat{u}[z^n]$ . Denoting by  $\hat{G}$  the CS of  $\hat{v}$ , we have

$$\hat{G} = 2 \sum_{n \geq 1} \hat{\nu}_{-n} z^n = i(\nu_0 - G),$$

$$\hat{G}_* = 2 \sum_{n \geq 1} \hat{\nu}_n z^{-n} = i(G_* - \nu_0),$$

and, consequently,

$$\hat{v} = \begin{cases} i(v - \text{leb } \nu_0) & (\text{in } \mathbb{P}), \\ i(\text{leb } \nu_0 - v) & (\text{in } \mathbb{P}_*). \end{cases}$$

Subtracting (1) and (2),

$$\frac{F - F_*}{2i} L = \frac{G + G_*}{2} \frac{M - M^{*p}}{2i} + \frac{G - G_*}{2} \frac{M + M^{*p}}{2i} + \frac{C - C^{*p}}{2i},$$

i.e.,

$$\hat{u}L = vM^- + \hat{v}M^+ + \text{leb } C^-, \quad (6)$$

where  $C^- = (C - C^{*p})/2i$  and

$$\hat{u} = \begin{cases} i(u - \text{leb } \mu_0) & (\text{in } \mathbb{P}), \\ i(\text{leb } \mu_0 - u) & (\text{in } \mathbb{P}_*), \end{cases}$$

with  $\mu_n = u[z^n]$ . The expressions (5) and (6) can be written in matrix form as

$$(u, \hat{u})L = (v, \hat{v}) \begin{pmatrix} M^+ & M^- \\ -M^- & M^+ \end{pmatrix} + \text{leb } (C^+, C^-).$$

The wild behaviour of the LST is originated by the presence of the functional  $\hat{v}$  and the Lebesgue functional in Equation 5. In other words, the relation between  $u$  and  $v$  fails to be a RM only due to the presence of the polynomial coefficients  $M^-$  and  $C^+$  in Equation 5. Therefore, a LST becomes a RM when  $M^- = C^+ = 0$ . This leads to the symmetry conditions which we already know that characterize those LST coming from a RM.

## 5 Generators of LST

In this section we analyze another aspect of the LST, namely, we will show how to generate them by composition of certain elementary LST that we will call generators.

First of all, without loss of generality, we will write any LST as

$$FA = GB + C, \quad A, B, C \in \mathbb{P}.$$

Then, we define the class  $(r, s, t)$  by the conditions  $\deg A = r$ ,  $\deg B = s$ ,  $\deg C = t$  and  $(A(0), B(0), C(0)) \neq (0, 0, 0)$ . If  $C = 0$  we will say that the corresponding LST belongs to the  $(r, s)$  class.

Note that the  $(0,0)$  class is generated by composition of LST belonging to the  $(0,0,0)$  class. The  $(1,0)$  and  $(0,1)$  classes generate by composition all the  $(r,s)$  classes with  $(r,s) \neq (0,0)$ . Analogously, the  $(1,0,0)$  and  $(0,1,0)$  classes generate the  $(r,s,0)$  classes with  $(r,s) \neq (0,0)$ . At the same time, the  $(1,0,0)$  and  $(0,1,0)$  classes are generated by the  $(0,0,0), (1,0)$  and  $(0,1)$  ones: any  $(1,0,0)$  class LST such as  $F(\alpha_0 + \alpha_1 z) = G\beta + c$  is the composition of  $F(\alpha_0 + \alpha_1 z) = \tilde{F}\alpha_0$  and  $\tilde{F}\alpha_0 = G\beta + c$ .

Consider now the elementary classes  $(0,0,0)$ ,  $(1,0)$  and  $(0,1)$ . We will show that they generate all the classes  $(r,s,t)$  with  $r,s,t \geq 0$ . For this purpose, it will be enough to prove that every  $(0,0,t)$  class LST, with  $t \geq 1$ , can be transformed into a  $(0,0,t_1)$  class LST, with  $t_1 < t$ , by means of elementary LST. This is because any LST in the  $(r,s,t)$  class can be reduced to a LST in the class  $(0,0,t')$  or  $(0,0)$  by  $(1,0,0)$  and  $(0,0,0)$  class LST, as it is easy to check.

Let

$$FA = GB + C \quad (7)$$

be a LST of class  $(r,s,t)$ . Let  $A_1 = A/A_0$  where  $A_0$  is a degree one divisor of  $A$ . The composition of  $FA_0 = \tilde{F} + c$  and  $\tilde{F}A_1 = GB + C - cA_1$  gives (7), the first LST being in the  $(1,0,0)$  class, and the second one in the  $(r-1,0,t')$  class with  $t' = \deg(C - cA) \leq \max\{r-1, t\}$ .

Now, we will see that every  $(0,0,t)$  class LST

$$F\alpha = G\beta + C, \quad t \geq 1,$$

is generated by the  $(0,0,0)$ ,  $(1,0)$  and  $(0,1)$  classes.

Let us consider the  $(1,0,0)$  and  $(1,0)$  class transformations

$$\tilde{F}z = \frac{F - \mu_0}{2\mu_{-k}}, \quad \tilde{G}z = \frac{G - \nu_0}{2\nu_{-j}},$$

where  $\mu_{-k}$  and  $\nu_{-j}$  are the first non zero coefficients of  $F$  and  $G$  with indices  $k, j \geq 1$ . From the equality  $F\alpha = G\beta + C$  we have

$$\alpha\mu_0 = \beta\nu_0 + C(0),$$

and from

$$(\tilde{F}2\mu_{-k}z + \mu_0)\alpha = (\tilde{G}2\nu_{-j}z + \nu_0)\beta + C$$

it follows that

$$\tilde{F}2\alpha\mu_{-k}z + \alpha\mu_0 = \tilde{G}2\beta\nu_{-j}z + \beta\nu_0 + C_1,$$

with  $C_1(z) = (C(z) - C(0))/z$  such that  $\deg C_1 \leq t-1$ .

This proves that the elementary LST classes generate by composition all the LST.

Finally, we will analyze the relation between the functionals related by the elementary classes.

A general  $(0,0,0)$  class LST has the form

$$F\alpha = G\beta + c,$$

where, without loss of generality, we can assume that  $\alpha \in \mathbb{R}$ , i.e.  $A = A^*$ . In such a case

$$u\alpha = v\operatorname{Re}(\beta) - \hat{v}\operatorname{Im}(\beta) + \operatorname{Re}(c),$$

in other words,

$$\begin{aligned} u\alpha &= v\operatorname{Re}(\beta) + v i \operatorname{Im}(\beta) - v_0 i \operatorname{Im}(\beta) + \operatorname{Re}(c) \quad (\text{in } \mathbb{P}), \\ u\alpha &= v\operatorname{Re}(\beta) - v i \operatorname{Im}(\beta) + v_0 i \operatorname{Im}(\beta) + \operatorname{Re}(c) \quad (\text{in } \mathbb{P}_*). \end{aligned}$$

Equivalently,

$$\begin{aligned} u\alpha &= v\beta - v_0 i \operatorname{Im}(\beta) + \operatorname{Re}(c) \quad (\text{in } \mathbb{P}), \\ u\alpha &= v\bar{\beta} + v_0 i \operatorname{Im}(\beta) + \operatorname{Re}(c) \quad (\text{in } \mathbb{P}_*), \end{aligned}$$

which describe completely the functional modification.

As for the  $(0, 1)$  class LST

$$F\alpha = GB, \quad B = \beta_0 + \beta_1 z, \quad \alpha \in \mathbb{R}^*,$$

we have that  $\alpha\mu_0 = \beta_0\nu_0$ , thus  $\beta_0 \in \mathbb{R}$ . Since

$$u\alpha = v \frac{B + B^*}{2} - \hat{v} \frac{B - B^*}{2i},$$

we find that

$$u\alpha = vB - \text{leb } i \frac{\beta_1 z - \bar{\beta}_1 z^{-1}}{2i} \nu_0 \quad (\text{in } \mathbb{P}),$$

$$u\alpha = vB_* - \text{leb } i \frac{\beta_1 z + \bar{\beta}_1 z^{-1}}{2i} \nu_0 \quad (\text{in } \mathbb{P}_*).$$

Analogously, for the  $(1, 0)$  class LST

$$F(\alpha_0 + \alpha_1 z) = G\beta, \quad \beta \in \mathbb{R}^*,$$

we conclude that  $\alpha_0 \in \mathbb{R}$  and

$$v\beta = uA - \text{leb } i \frac{\alpha_1 z - \bar{\alpha}_1 z^{-1}}{2i} \mu_0 \quad (\text{in } \mathbb{P}),$$

$$v\beta = uA_* + \text{leb } i \frac{\alpha_1 z - \bar{\alpha}_1 z^{-1}}{2i} \mu_0 \quad (\text{in } \mathbb{P}_*),$$

where  $A = \alpha_0 + \alpha_1 z$ .

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